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Differential Subordination and Superordination of Analytic Functions Defined By Cho - Kwon - Srivastava Operator

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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS DEFINED BY CHO - KWON - SRIVASTAVA OPERATOR

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Differential Subordination and Superordination of Analytic Functions Defined By Cho-Kwon-Srivastava Operator

Jamal M. Shenan

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I. Introduction

Let $H(U)$ be the class of functions analytic in $U=\{z : z \in C \text{ and } |z|<1\}$ and $H[a,n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z)=a+a_n z^n+a_{n+1}z^{n+1}+...,$ With $H_0 \equiv H[0,1]$ and $H \equiv H\ [1,1]$. Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \left(p \in \mathrm{N}=\{1,2,3,...\}; z \in U\right), \tag{1.1}$$

and let $A(1)=A$. Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z \in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if F is univalent, then $f(z) \prec F(z)$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$ (see [1] and [2]).

For two functions $f(z)$ given by (1.1) and

$$g(z)=z^p+\sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \tag{1.2}$$

The hadmard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}=(g*f)(z).$$

Saitoh [8] introduce a linear operator:

$$L_p(a,c): A_p \to A_p$$

defined by

$$L_p(a,c)=\phi_p(a,c;z)*f(z) \quad \left(z \in U\right), \tag{1.3}$$

where

$$\phi_p(a,c;z)=\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},$$

and $(a)_k$ is the Pochammer symbol. In 2004, Cho, Kwon and Srivastava [4] introduced the linear operator $L_p^{\lambda}(a,c): A_p \to A_p$ analogous to $L_p(a,c)$ defined by

$$L_p^{\lambda}(a,c)f(z)=\phi_p^{\lambda}(a,c;z)*f(z) \quad \left(z \in U; a,c \in R \setminus Z_0^-; \lambda>-p\right), \tag{1.4}$$

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where $\phi_p^{\lambda}(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition :

$$\phi_p(a,c;z)*\phi_p^{\lambda}(a,c;z)=\frac{z^p}{(1-z)^{\lambda+p}} \tag{1.5}$$

We can easily find from (1.4) and (1.5) and for the function $f(z)\in A_p$ that

$$L_p^{\lambda}(a,c)f(z)=z^p+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{k!(a)_k}a_{k+p}z^{k+p} \quad . \tag{1.6}$$

It is easily verified from (1.6) that

$$z\left(L_p^{\lambda}(a+1,c)f\right)'(z)=aL_p^{\lambda}(a,c)f(z)-(a-p)L_p^{\lambda}(a+1,c)f(z) \quad , \tag{1.7}$$

and

$$z\left(L_p^{\lambda}(a,c)f\right)'(z)=(\lambda+p)L_p^{\lambda+1}(a,c)f(z)-\lambda L_p^{\lambda}(a,c)f(z). \tag{1.8}$$

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To prove our results, we need the following definitions and lemmas.

Denote by Q the set of all functions $q(z)$ that are analytic and injective on $\overline{U}/E(q)$ where

$$E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq$ for $\zeta\in\partial U/E(q)$. Further let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

Definition 1 ([6]). let Ω be a set in C, $q\in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consist of those functions $\psi:C^3\times U\to C$ that satisfy the admissibility condition:

$$\psi(r,s,t;z)\notin\Omega$$

whenever

$$r=q(\zeta),\ s=k\zeta q'(\zeta),\ R\left\{\frac{t}{s}+1\right\}\geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U/E(q)$ and $k\geq n$ We write $_1[\Omega,q]$ as $\Psi[\Omega,q]$.

Definition 2 ([7]). let Ω be a set in C, $q(z)\in H[a,n]$ with $q'(z)\neq 0$ The class of admissible functions $\Psi_n'[\Omega,q]$ consist of those functions $\psi:C^3\times\overline{U}\to C$ that satisfy the admissibility condition

$$\psi(r,s,t;\zeta)\notin\Omega$$

whenever

$$r=q(z),\ s=\frac{zq'(z)}{m},\ R\left\{\frac{t}{s}+1\right\}\geq\frac{1}{m}R\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U$ and $m\geq n\geq 1$. In particular , we write $\Psi_1'[\Omega,q]$ as $\Psi'[\Omega,q]$.

Lemma 1 ([6]). Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If the analytic function $p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$ satisfies

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega,$$

then

$$p(z)\prec q(z).$$

Lemma 2 ([7]). Let $\psi\in\Psi_n'[\Omega,q]$ with $q(0)=a$. If $p(z)\in Q(a)$ and $\psi(p(z),zp'(z),z^2p''(z);z)$ is univalent in U then

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z);z):z\in U\},$$

implies

$$q(z) \prec p(z).$$

In the present investigation, the differential subordination result of Miller and Mocanu [6,7] is extended for analytic functions in the open unit disk , which are associated with Cho - Kwon - Srivastava operator $L_p^{\lambda}(a,c)$ $\left(a,c \in R \setminus Z_0^-; \lambda > -p\right)$, and we obtain certain other related results . A simililar problem for analytic functions was Srivastava [5], Aouf and Seoudy [3], Aghalary et al. [1], Ali et al. [2]. Additionally, the corresponding differential superordination problem is investigated, and several sandwichtype result are obtained.

II. Subordination Results Involving the Cho-Kwon–Srivastava Operator

$$I_p^{\lambda}(a,c)f(z).$$

Definition 3. Let Ω be a set in C and $q(z) \in Q_0 \cap H[0,p]$. The class of admissible functions $\Phi_I[\Omega,q]$ consist of those functions $\phi: C^3 \times U \to C$: that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + (a-p)q(\zeta)}{a},$$

$$R\left\{\frac{a(a-1)w - (a-p)(a-p-1)u}{av - (a-p)u} - 2(a-p) + 1\right\} \geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Theorem 1. Let $\phi \in \Phi_I[\Omega,q]$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) : z \in U\right\} \subset \Omega, \tag{2.1}$$

then

$$I_p^{\lambda}(a+1,c)f(z) \prec q(z)$$

$$\left(z \in U;\ a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = I_p^{\lambda}(a+1,c)f(z) \qquad \left(z \in U;\ a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right). \tag{2.2}$$

In view of the relation (1.7) from (2.2), we get

$$I_p^{\lambda}(a,c)f(z) = \frac{zp'(z) + (a-p)p(z)}{a}. \tag{2.3}$$

Further computation show that

$$I_p^{\lambda}(a+1,c)f(z) = \frac{\left[z^2p''(z) + 2(a-p)zp'(z) + (a-p)(a-p-1)p(z)\right]}{a(a-1)}. \tag{2.4}$$

Define the transformation from C^3 to C by

$$u = r,\ v = \frac{s + (a-p)r}{a},\quad w = \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+(a-p)r}{a}, \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}; z\right). \tag{2.5}$$

The proof shall make use of Lemma 1. Using equation (2.2) , (2.3) and (2.4), then from (2.5) , we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right). \tag{2.6}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega .$$

The proof is completed if it can be shown that the admissibility condition for is equivalent to the admissibility condition for as given in Definition 1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{a(a-1)w-(a-p)(a-p-1)u}{av-(a-p)u}-2(\lambda+p)+1\right\},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1,

$$p(z) \prec q(z) \text{ or } I_p^{\lambda}(a+1,c)f(z) \prec q(z).$$

If $\Omega \neq C$ is a simply connected domain , then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h, q]$.

The following result is an immediate consequence of Theorem 1.

Theorem 2. Let $\phi \in \Phi_I[h,q]$, If $f(z) \in A(p)$ satisfies

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \prec h(z), \tag{2.7}$$

then

$$I_p^{\lambda}(a+1,c)f(z) \prec q(z),$$

where $\left(p \in N; a,c \in R \setminus Z_0^-; \lambda > -p; z \in U\right)$.

Our next result is an extension of Theorem1 to the case where the behavior of $q(z)$, on ∂U is not known.

Corollary 1. Let $\Omega \subset C$ and let $q(z)$,be univalent in U, $q(0)=0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in A(p)$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \in \Omega, \tag{2.8}$$

then

$$I_p^{\lambda}(a+1,c)f(z) \prec q(z)$$

$$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right).$$

Proof. Theorem 1 yields $I_p^{\lambda}(a+1,c)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

If $q(z) = Mz, M > 0$, and in view of Definition 1, The class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$ is described below .

Definition 4. let Ω be a set in C and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consist of those functions $\phi : C^3 \times U \to C$ such that

$$\phi\left(Me^{i\theta}, \frac{k+(a-p)}{a}Me^{i\theta}, \frac{L+\left[2(a-p-1)k+(a-p-1)(a-p-2)\right]Me^{i\theta}}{a(a-1)}; z\right) \notin \Omega \tag{2.9}$$

whenever $,z \in U, \theta \in R, R\left\{Le^{-i\theta}\right\} \geq (k-1)k\mathrm{M}$ for all real θ, $p \in \mathrm{N}$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Corollary 2. Let $\phi \in \Phi_I[\Omega, M]$. If $f(z) \in A(p)$ satisfies

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \in \Omega ,$$

then

$$\left|I_p^{\lambda}(a+1,c)f(z)\right| < M . \qquad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \left\{w : |w| < M\right\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$, then the corollary (2.2) takes the following form .

Corollary 3. Let $\phi \in \Phi_I[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(I_p^\lambda(a+1,c)f(z), I_p^\lambda(a,c)f(z), I_p^\lambda(a-1,c)f(z); z\right)\right| < M \quad ,$$

then

$$\left|I_p^\lambda(a+1,c)f(z)\right| < M \,. \qquad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

Now , we introduce a new class of admissible functions $\Phi_{I,1}[\Omega, q]$.

Definition 5. Let Ω be a set in C, $q \in Q_0 \cap H_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\; v = \frac{k\zeta q'(\zeta) + (a-1)q(\zeta)}{a},$$

$$R\left\{\frac{(a-1)\left[aw - (a-2)u\right]}{av - (a-1)u} - 2(a-p) + 3\right\} \ge kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \ge p$.

Theorem 3. Let $\phi \in \Phi_{I,1}[\Omega, q$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega \quad , \qquad (2.10)$$

then

$$\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}} \prec q(z)\,.$$

$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$.

Proof . Define the analytic function $p(z)$ in U by

$$p(z) = \frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}} \qquad (2.11)$$

In the view of relation (1.7) and from (2.11) we get ,

$$\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}} = \frac{zp'(z) + (a-1)p(z)}{a} \quad . \qquad (2.12)$$

Further computation show that

$$\frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}} = \frac{\left[z^2p''(z) + 2(a-1)zp'(z) + (a-1)(a-2)p(z)\right]}{a(a-1)} \quad . \qquad (2.13)$$

Define the transformation from C^3 to C by

$$u = r,\; v = \frac{s + (a-1)r}{a} \;,\; w = \frac{t + 2(a-1)s + (a-1)(a-2)r}{a(a-1)} \quad . \qquad (2.14)$$

Let

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$$\psi(r,s,t;z)=\phi(u,v,w;z)$$
$$=\phi\left(r,\frac{s+(a-1)r}{a},\frac{t+2(a-1)s+(a-1)(a-2)r}{a(a-1)};z\right) \qquad (2.15)$$

The proof shall make use of Lemma 1. Using equation (2.11), (2.12) and (2.13), from (2.15), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right) . \qquad (2.16)$$

Hence (2.10) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega .$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{I,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{(a-1)\left[aw-(a-2)u\right]}{av-(a-1)u}-2(a-p)+3\right\},$$

and hence $\psi\in\Psi_p[\Omega,q]$. By Lemma 1 , $p(z)\prec q(z)$ or

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\prec q(z) .$$

If $\Omega\neq C$ is a simply connected domain , then $\Omega=h(U)$, for some conformal mapping $h(z)$ of U onto Ω

In this case the class $\Phi_{I,1}[h(U),q]$ is written as $\Phi_{I,1}[h,q]$.

The following result is an immediate consequence of Theorem 3.

Theorem 4. Let $\phi\in\Phi_{I,1}[\Omega,q]$, If $f(z)\in A(p)$ satisfies

$$\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right)\prec h(z), \qquad (2.17)$$

then

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\prec q(z) .$$

$$\left(p\in N;\lambda>-p;a,c\in R\setminus Z_0^-;z\in U\right).$$

If $q(z)=Mz\,,M>0$, The class of admissible functions $\Phi_{I,1}[\Omega,q]$, denoted by $\Phi_{I,1}[\Omega,M]$, is described below .

Definition 6. let Ω be a set in C and $M>0$. The class of admissible functions $\Phi_{I,1}[\Omega,q]$, consists of those functions $\phi:C^3\times U\to C$ such that

$$\phi\left(Me^{i\theta},\frac{k+a-1}{a}Me^{i\theta},\frac{L+(a-1)\{2k+(a-2)\}Me^{i\theta}}{a(a-1)};z\right)\notin\Omega , \qquad (2.18)$$

whenever

$$z\in U,\theta\in R,R\left\{Le^{-i\theta}\right\}\geq(k-1)k\mathrm{M} \text{ for all real } \theta, p\in\mathrm{N} \text{ and } a\in R\setminus Z_0^-,k\geq p .$$

Corollary 4. Let $\phi \in \Phi_{I,1}[\Omega, M]$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right) \in \Omega ,$$

then

$$\left|\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right| < M \ .\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$, then the previous Corollary 4 takes the following form .

Corollary 5. Let $\phi \in \Phi_{I,1}[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right)\right| < M ,$$

then

$$\left|\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right| < M \ .\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

Next , we introduce a new class of admissible functions $\Phi_{I,2}[\Omega, q]$.

Definition7. Let Ω be a set in C, $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_{I,2}[\Omega, q]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{a-1}\left\{-1 + aq(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right\},$$

$$R\left\{\frac{\{(a-2)w - (a-1)v + 1\}}{(a-1)v - au + 1} - 2au + (a-1)v - 1\right\} \geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Theorem 5. Let $\phi \in \Phi_{I,2}[\Omega, q]$ and $I_p^\lambda(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right) : z \in U\right\} \subset \Omega \quad , \qquad (2.19)$$

then

$$\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \prec q(z).$$

$$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right).$$

Proof . Define the analytic function $p(z)$ in U by

$$p(z) = \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} . \qquad (2.20)$$

Using (2.20) , we get

$$\frac{zp'(z)}{p(z)}=\frac{z(I_p^{\lambda}(a,c)f(z))'}{(I_p^{\lambda}(a,c)f(z))}-\frac{z(I_p^{\lambda}(a+1,c)f(z))'}{(I_p^{\lambda}(a+1,c)f(z))}\quad. \tag{2.21}$$

By making use of the relation (1.7) in (2.21) , we get

$$\frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)}=\frac{1}{(a-1)}\left\{-1++ap(z)+\frac{zp'(z)}{p(z)}\right\} \tag{2.22}$$

Further computation show that

$$\frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)}=\frac{1}{(a-1)}$$

$$\left[-2+\frac{zp'(z)}{p(z)}+ap(z)+\frac{\frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+azp'(z)}{\frac{zp'(z)}{p(z)}+ap(z)-1}\right]\quad. \tag{2.23}$$

Define the transformation from C^3 to C by

$$u=r\,,v=\frac{1}{(a-1)}\left\{-1+ar+\frac{s}{r}\right\},\ w=\frac{1}{(a-1)}\left\{-2+\frac{s}{r}+ar+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+as}{\frac{s}{r}+ar-1}\right\}\quad. \tag{2.24}$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$

$$=\phi\left(r,\frac{1}{(a-1)}\left\{-1+ar+\frac{s}{r}\right\},\frac{1}{(a-1)}\left\{-2+\frac{s}{r}+ar+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+as}{\frac{s}{r}+ar-1}\right\};z\right) \tag{2.25}$$

The proof shall make use of Lemma 1. Using equation (2.20), (2.22) and (2.23), then (2.25), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)},\frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)},\frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)};z\right) \tag{2.26}$$

Hence (2.19) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega\,.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{I,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{v\,\{(a-2)w-(a-1)v+1\}}{(a-1)v-au+1}-2\,au+(\,a-1)\,v+1\right\},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1, $p(z) \prec q(z)$ or $\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \prec q(z)$.

If $\Omega \neq C$ is a simply connected domain , then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{I,2}[h(U), q]$ is written as $\Phi_{I,2}[h, q]$.

The following result is an immediate consequence of Theorem 5.

Theorem 6. Let $\phi \in \Phi_{I,2}[h,q]$ and $I_p^\lambda(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right) \prec h \qquad (2.27)$$

then

$$\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \prec q(z).$$

$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$.

If $q(z) = Mz$, $M > 0$, The class of admissible functions $\Phi_{I,2}[\Omega, q]$, denoted by $\Phi_{I,2}[\Omega, M]$, is described below .

Definition 8. let Ω be a set in C and $M > 0$. The class of admissible functions $\Phi_{I,2}[\Omega, M]$ consist of those functions $\phi : C^3 \times U \to C$ such that

$$\phi\left(Me^{i\theta}, \frac{1}{(a-1)}\left(k-1+aMe^{i\theta}\right), \frac{1}{(a-1)}\left\{k-2+aMe^{i\theta} + \frac{Le^{-i\theta}+kM+akM^2-k^2M}{M(k-1)+aM^2e^{i\theta}}\right\}; z\right) \notin \Omega \qquad ,(2.28)$$

whenever

$z \in U, \theta \in R, \Re\left\{Le^{-i\theta}\right\} \geq (k-1)k\text{M}$ for all real θ, $p \in \text{N}$; $a \in R \setminus Z_0^-$ and $k \geq p$.

Corollary 6. Let $\phi \in \Phi_{I,2}[\Omega, M]$ and $I_p^\lambda(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}\right| < M, \qquad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_{I,2}[\Omega, M]$ is simply denoted by $\Phi_{I,2}[M]$, then Corollary 6 takes the following form .

Corollary 7. Let $\phi \in \Phi_{I,2}[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right)\right| < M,$$

then

$$\left|\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}\right| < M, \quad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

III. Superordination of the Cho-Kwon–Srivastava Operator $I_p^{\lambda}(a,c)f(z)$

The dual problem of the differential subordination, that is, differential superordination of the operator $I_p^{\lambda}(a,c)f(z)$ is investigated in this section . For this purpose the class of the admissible functions is given in the following definition.

Definition 9. let Ω be a set in $q(z) \in H[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_I[\Omega,q]$ consists of those functions $\phi: C^3 \times \overline{U} \to C$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(z),\ v = \frac{zq'(z) + m(a-p)q(z)}{ma},$$

$$R\left\{\frac{a(a-1)w - (a-p)(a-p-1)u}{av - (a-p)u} - 2(a-p) + 1\right\} \leq \frac{1}{m} R\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U$, $a \in R \setminus Z_0^-, z \in U$ and $m \geq p$.

Theorem 7. Let $\phi \in \Phi'_I[\Omega,q]$. If $f(z) \in A(p)$, $I_p^{\lambda}(a,c)f(z) \in Q_0$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right): z \in U\right\} \tag{3.1}$$

implies

$$q(z) \prec I_p^{\lambda}(a+1,c)f(z).$$

$$\left(z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right).$$

Proof. From (2.6) and (3.1),we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_I[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'_p[\Omega,q]$, and by

Lemma 2, $q(z) \prec p(z)$ or $q(z) \prec I_p^{\lambda}(a+1,c)f(z)$.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω In this case the class $\Phi'_I[h(U),q]$ is written as $\Phi'_I[h,q]$.

The following result is an immediate consequence of Theorem 7.

Theorem 8. Let $h(z)$ be analytic on U and $\phi \in \Phi'_I[h,q]$. If $f(z) \in A(p)$, $I_p^{\lambda}(a+1,c)f(z) \in Q_0$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right)$$

is univalent in U,

then

$$h(z)\prec\phi\left(I_p^{\lambda}(a+1,c)f(z),I_p^{\lambda}(a,c)f(z),I_p^{\lambda}(a-1,c)f(z);z\right), \tag{3.2}$$

implies

$$q(z)\prec I_p^{\lambda}(a+1,c)f(z).$$

Now, we introduce a new class of admissible functions $\Phi'_{I,1}[\Omega,q]$.

Definition 9. Let Ω be a set in $C, q(z)\in H_0$ with $zq'(z)\neq 0$. The class of admissible functions $\Phi'_{I,1}[\Omega,q]$ consists of those functions $\phi: C^3\times\overline{U}\to C$ that satisfy the admissibility condition :

$$\phi(u,v,w;\zeta)\in\Omega$$

whenever

$$u=q(z),\ v=\frac{zq'(z)+m(a-1)q(z)}{ma},$$

$$R\left\{\frac{a(a-1)w-(a-2)u}{av-(a-1)u}-2(a-p)+3\right\}\leq\frac{1}{m}R\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U$ and $m\geq p$.

Now, we will give the dual result of Theorem 3 for differential superordination.

Theorem 9. Let $\phi\in\Phi'_{I,1}[\Omega,q]$. If $f(z)\in A(p)$, $\dfrac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\in Q_0$ and

$$\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right)$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right):z\in U\right\} \tag{3.3}$$

implies

$$q(z)\prec\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}.$$

$$\left(z\in U;\ a,c\in R\setminus Z_0^-;\lambda>-p;p\in N\right).$$

Proof. From (2.16) and (3.3), we have

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z);z):z\in U\}.$$

From (2.12), we see that the admissibility condition for $\phi\in\Phi'_{I,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi\in\Psi'[\Omega,q]$ and by

Lemma 2, $q(z)\prec p(z$ or $q(z)\prec\dfrac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}$.

If $\Omega\neq C$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{I,1}[h(U),q]$ is written as $\Phi'_{I,1}[h,q]$.

The following result is an immediate consequence of Theorem 9.

Theorem 10. Let $q(z) \in H_0$, $h(z)$ be univalent in U and $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f(z) \in A(p)$, $\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}} \in Q_0$ and

$$\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right) \qquad (3.4)$$

implies

$$q(z) \prec \frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}.$$

$$\left(z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right)$$

Finally, we introduce down a new class of admissible functions $\Phi'_{I,2}[\Omega, q]$.

Definition 10. let Ω be a set in C, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in H$. The class of admissible functions $\Phi'_{I,2}[\Omega, q]$ consists of those functions $\phi : C^3 \times \overline{U} \to C$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{a-1}\left\{-1 + aq(z) + \frac{zq'(z)}{mq(z)}\right\},$$

$$R\left\{\frac{\{(a-2)w - (a-1)v + 1\}v}{(a-1)v - au + 1} + (a-1)u - 2av + 1\right\} \leq \frac{1}{m} R\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Now , we will give the dual result of theorem 5.

Theorem 11. Let $\phi \in \Phi'_{I,2}[\Omega, q$. If $f(z) \in A(p)$, $\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \in Q_1$ and

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right)$$

is univalent in U, then

$$\Omega \subset \phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right), \qquad (3.5)$$

implies

$$q(z) \prec \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}.$$

$\left(z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right)$.

Proof. From (2.26) and (3.5) , we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.$$

From (2.24) , we see that the admissibility condition for $\phi \in \Phi'_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by lemma 2, $q(z) \prec p(z)$ or $q(z) \prec \dfrac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}$.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{I,2}[h(U), q]$ is written as $\Phi'_{I,2}[h, q]$.

The following result is an immediate consequence of Theorem 11.

Theorem 12. Let $h(z)$ be analytic in U and $\phi \in \Phi'_{I,2}[h, q]$. If $f(z) \in A(p)$, $\dfrac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)} \in Q_1$ and

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right)$$

is univalent in U , then

$$h(z) \prec \phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right), \qquad (3.6)$$

implies

$$q(z) \prec \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}.$$

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